Tobler’s Law

- “All places are related, but nearby places are related more than distant places”
  - Corollary: fields vary smoothly, slowly and show strong “spatial autocorrelation” – attribute(s) and location are strongly correlated \( z_i = f(x_i, y_i) \)

Spatial Interpolation

- Determination of unknown values or attributes on the basis of values nearby
  - Used for data that define continuous fields
    - E.g. temperature, rainfall, elevation, concentrations
    - Contouring, raster resampling are applications already discussed

Spatial Interpolation = Spatial Prediction

E.g. Interpolate between variably spaced data to create uniform grid of values
Interpolation Methods

- All address the meaning of “near” in Tobler’s law differently
  - How does space make a difference?
  - Statistical mean not best predictor if Tobler’s law is true

Interpolation methods

- Inverse Distance Weighting (IDW)
  - Assumes influence of adjacent points decreases with distance
  
  \[
  z_0 = \frac{\sum_{i=1}^{n} w_i z_i}{\sum_{i=1}^{n} w_i}
  \]

  Where:
  - \( z_0 \) = value of estimation point
  - \( z_i \) = value of neighboring point
  - \( w_i \) = weighting factor; e.g. \( 1/(\text{distance from neighbor})^2 \)

Inverse Distance Weighting

On basis of four nearest neighbors:

\[
z_0 = \frac{8/(1)^2 + 8/(2)^2 + 6/(2.5)^2 + 5/(2)^2}{(1.66)}
\]

\[
z_0 = \frac{8.0 + 2.0 + 0.96 + 1.25}{(1.66)} = 7.36
\]

I.D.W.

- Unknown value is the average of the observed values, weighted by inverse of distance, squared
  - Distance to point doubles, weight decreases by factor of 4

  Can alter IDW by:
  - Alter number of closest points
  - Choose points by distance/search radius
  - Weight be directional sectors
  - Alter distance weighting; e.g. cube instead of square
I.D.W. Characteristics

- Is an exact method of interpolation – will return a measured value when applied to measured point.
  - Will not generate smoothness or account for trends, unlike methods that are “inexact”
- Weights never negative → interpolated values can never be less than smallest z or greater than largest z. “Peaks” and “pits” will never be represented.

Interpolation Methods

- IDW is inappropriate for values that don’t decrease as a function of distance (e.g. topography)
- Other deterministic techniques:
  - Spline
  - Trend

Exact Methods - Spline

- Fit minimum curvature surface through observation points; interpolate value from surface
- Good for gently varying surfaces
  - E.g. topography, water table heights
- Not good for fitting large changes over short distances
- Surface is allowed to exceed highest and be less than lowest measured values
Spatial Interpolation - Geostatistics

**Exact Methods: IDW vs. Spline**

- **IDW:**
  - No predicted highs or lows above max. or min. values
  - No smoothing; surface can be rough

- **Spline:**
  - Minimum curvature result good for producing smooth surfaces
  - Can’t predict large changes over short distances

*(Images from ArcGIS 9.2 Help Files)*

**Comparisons - IDW vs. Spline**

- Note smoothing of Spline – less “spikey”
- IDW contours less continuous, fewer inferred maxima and minima

**Inexact (Approximate) Methods**

- Trend surface – curve fitting by least squares regression
- Kriging – weight by distance, consider trends in data

**Approximate Methods - Trend**

- Fits a polynomial to input points using least squares regression.
- Resulting surfaces minimize variance w.r.t. input values, i.e. sum of difference between actual and estimated values for all inputs is minimized.
- Surface rarely goes through actual points
- Surface may be based on all data (“Global” fit) or small neighborhoods of data (“Local” fits).
Trend Surfaces

Equations are either:
- Linear – 1st Order: fit a plane
  - \( Z = a + bX + cY \)
- Quadratic – 2nd Order: fit a plane with one bend - parabolic
  - \( Z = (1^{st} \text{ Order}) + dX^2 + eXY + fY^2 \)
- Cubic – 3rd Order: fit a plane with 2 bends - hyperbolic
  - \( Z = (2^{nd} \text{ Order}) + gX^3 + hX^2Y + iXY^2 + Y^3 \)

Where:
- \( a, b, c, d, \text{ etc.} \) = constants derived from solution of simultaneous equations
- \( X, Y \) = geographic coordinates

Trend Surfaces – “Global Fitting”

- Linear (Plane)
- Quadratic (Parabolic surface)
- Cubic (Hyperbolic surface)

Source: Burrough, 1986

Contour maps of trend surfaces

Trend Surfaces – Local fitting

- Local Polynomial interpolation fits many polynomials, each within specified, overlapping “neighborhoods”.
- Neighborhood surface fitting is iterative; final solution is based on minimizing RMS error
- Final surface is composed of best fits to all neighborhoods
- Can be accomplished with tool in ESRI Geostatistical Analyst extension

Step 1

2-D profile view of a model surface
- Neighborhood 1 points (red) are being fit to a plane by iteration (2 steps are shown) and an interpolated point is being created
Step 2
Trend Surfaces – Local fitting 2
- Model surface generated by many local fits
  - Note that several neighborhoods share some of the same data points: neighborhoods overlap

Step 3
Trend Surfaces – Local fitting 3
- Five different polynomials generate five local fits; in this example all are 1st Order.

Step 4
Trend Surfaces – Local fitting 4
- No strong reason to assume that z correlated with x, y in these simple ways
- Fitted surface doesn’t pass through all points in Trend
- Data aren’t used to help select model
- Exploratory, deterministic techniques, but theoretically weak

Step 5
Trend Surfaces – Local fitting 5
- Original data points are black
- Interpolated points are in colors
- Note that model surface (purple) passes through interpolated points, not measured data points.
Deterministic vs. Geostatistical Models

- Deterministic: purely a function of distance
  - No associated uncertainties are used or derived
  - E.g. IDW, Trend, Spline
- Geostatistical: based on statistical properties
  - Uncertainties incorporated and provided as a result
  - Kriging

Approximate Methods - Kriging

- Kriging
  - Another inverse distance method
  - Considers distance, cluster and spatial covariance (autocorrelation) – look for patterns in data
  - Fit function to selected points; look at correlation, covariance and/or other statistical parameters to arrive at weights – interactive process
  - Good for data that are spatially or directionally correlated (e.g. element concentrations)

Kriging

- Look for patterns over distances, then apply weights accordingly.
- Steps:
  1) Make a description of the spatial variation of the data - variogram
  2) Summarize variation by a function
  3) Use this model to determine interpolation weights

Kriging – Step 1

- Describe spatial variation with Semivariogram

(Point cloud)
### Spatial Interpolation - Geostatistics

**Kriging – Step 1**
- Divide range into series of “lags” (“buckets”, “bins”)
- Find mean values of lags

\[
\frac{(Z_i - Z_j)^2}{2}
\]

Distance between pairs of points

**Kriging – Step 2**
- Summarize spatial variation with a function
  - Several choices possible; curve fitting defines different types of Kriging (circular, spherical, exponential, gaussian, etc.)

\[
\frac{(Z_i - Z_j)^2}{2}
\]

Distance between pairs of points

**Key features of fitted variogram:**
- **Nugget**: semivariance at \(d = 0\)
- **Range**: \(d\) at which semivariance is constant
- **Sill**: constant semivariance beyond the range

**Nugget**

**Range**

**Sill**

\[\text{Distance (d)}\]
Kriging – Step 2
- Model surface profiles and their variograms:
  - As local variation in surface increases, range decreases, nugget increases

Source: O’Sullivan and Unwin, 2003

Kriging – Step 3
- Determine Interpolated weights
  - Use fitted curve to arrive at weights – not explained here; see O’Sullivan and Unwin, 2003 for explanation
  - In general, nearby values are given greater weight (like IDW), but direction can be important (e.g. “shielding” can be considered)

Review:
Deterministic vs. Geostatistical Models
- **Deterministic**: interpolation purely a function of distance
  - No associated uncertainties are used or derived
  - E.g. IDW, Trend, Spline
- **Geostatistical**: interpolation is statistically based
  - Uncertainties incorporated and provided as a result
  - Kriging

Kriging – Part II
- **Goal**: predict values where no data have been collected
- Relies on first establishing:
  - **DEPENDENCY** – z is, in fact, correlated with distance
  - **STATIONARITY** – z values are stochastic (except for spatial dependency they are randomly distributed) and have no other dependence – use “detrending” or transformation tools if not Gaussian
  - **DISTRIBUTION** – works best if data are Gaussian. If not they have to first be made close to Gaussian.
ESRI Geostatistical Analyst Products

- **Map types:**
  - Prediction – contours of interpolated values
  - Prediction Standard Errors – show error distribution, as quantified by minimized RMS error (see below)
  - Probability – show where values exceed a specified threshold
  - Quantile – show where thresholds overestimate or underestimate predictions

Maps: (e.g. max. ozone concentration, 1999)

- **ESRI Geostatistical Analyst Products**

Some Kriging Products

- **Prediction map** – interpolated values
- **Probability map** – showing where critical values exceeded

Kriging – Part II

- **Goal:** predict values where no data have been collected
- Relies on first establishing:
  - **DEPENDENCY** – z is, in fact, correlated with distance
  - **STATIONARITY** – z values are stochastic (except for spatial dependency they are randomly distributed) and have no other dependence – use “detrending” or transformation tools if not Gaussian
  - **DISTRIBUTION** – works best if data are Gaussian. If not they have to first be made close to Gaussian.
1. SPATIAL DEPENDENCY

- Test with semivariogram & cross-validation plots

Spatial Dependence: Semivariogram

Semivariogram and temperature distribution map of winter temperature data for the USA.

Figures from ESRI “Using Geostatistical Analyst”

Spatial Dependence: Cross-Validation Diagnostic

- Use a subset of the data to test measured vs. predicted values

Kriging – Part II

- **Goal:** predict values where no data have been collected
- Relies on first establishing:
  - **DEPENDENCY** – z is, in fact, correlated with distance
  - **STATIONARY** – z values are stochastic (except for spatial dependency they are randomly distributed) and have no other dependence – use “detrending” or transformation tools if not Gaussian
  - **DISTRIBUTION** – works best if data are Gaussian. If not they have to first be made close to Gaussian.
2. STATIONARITY - Randomness

- Data variance and mean is the same at all localities (or within a neighborhood of nearest points); data variance is constant in the neighborhood of investigation.
- Correlation (covariance) depends only on the vector that separates localities, not exact locations, number of measurement or direction.

California Ozone Demo.

- Data in “Geostat_demo” folder

ArcGIS Kriging Processing Steps

1. Add and display the data
2. Explore the data’s statistical properties
3. Select a model to create a surface – make a prediction map!
4. Assess the result
5. Compare to other models

Data Exploration

1. Examine the distribution – normal (Gaussian)? Transformation to normal required?
   - Histograms and QQPlots
2. Identify trends, if any
   - Trend Analysis
3. Understand spatial autocorrelation and directional influences
   - Semivariogram analysis
Data Exploration:
Examine the Distribution

- Normal (Gaussian) distribution? Transformation to normal required?
  - Histogram tool, QQPlot tool (compare real and standard normal distributions)

Data Exploration:
Identify Trends, If Any

- Underlying trends affect Kriging assumption of randomness – remove and work with “residuals”
  - Trend Analysis tool

Data Exploration:
Spatial Autocorrelation & Directional Influences

- Variogram Analysis:
  - Look for correlation with distance
  - Look for directional trends among pairs of points
  - Semivariogram/ Covariance Cloud tool

ArcGIS Kriging Processing Steps

1. Add and display the data
2. Explore the data’s statistical properties
3. Select a model to create a surface – make a prediction map!
4. Assess the result
5. Compare to other models
Mapping Ozone Concentration

1. Incorporate results of Data Exploration into Model selection
   - This example:
     - remove underlying trends discovered during data exploration *that have a rational explanation*. (Analysis is then performed on residuals and trend surface is added back into final surface) = "Detrending"
     - Remove directional trends between pairs of points – in certain directions closer points are more alike than in other directions = "anisotropy removal"

Mapping Ozone Concentration – Interpolation & Cross Validation

2. Define search neighborhood for interpolation (c.f. I.D.W.)
   - Use a search ellipse (or circle) to find nearest neighbors; specify radii of ellipse, min. & max. number of points per sectors

3. Examine Cross Validation plot
   - Predicted vs. Measured for subset(s) of the data
     - "Mean error" should be close to zero
     - "RMS error" and "mean standardized error" should be small
     - "RMS standardized error" should be close to one.

ArcGIS Kriging Processing Steps

1. Add and display the data
2. Explore the data’s statistical properties
3. Select a model to create a surface – make a prediction map!
4. Assess the result – Cross Validation Plots
5. Compare to other models

Comparing Model Results

Cross validation comparisons:

- "Mean error" should be close to zero
- "RMS error" and "mean standardized error" should be small
- "RMS standardized error" should be close to one.
Probality Mapping with Indicator Kriging

- Task: Make a map that show the probability of exceeding a critical threshold, e.g. 0.12 ppm ozone for an 8 hr. period
- Technique:
  - Transform data to a series of 0s and 1s according to whether they are above or below the threshold
  - Use a semivariogram on transformed data; interpret indicator prediction values as probabilities