Spatial Interpolation & Geostatistics

\[(Z_i - Z_j)^2 / 2\]

Distance between pairs of points
Tobler’s Law

“All places are related, but nearby places are related more than distant places”

Corollary: fields vary smoothly, slowly and show strong “spatial autocorrelation” – attribute(s) and location are strongly correlated: \( Z_i = f(x_i, y_i) \)
Spatial Interpolation

- Determination of unknown values or attributes on the basis of values nearby
  - Used for data that define continuous fields
    - E.g. temperature, rainfall, elevation, concentrations
    - Contouring, raster resampling are applications already discussed

Spatial Interpolation = Spatial Prediction
Spatial Interpolation

- I.e. Interpolate between variably spaced data to create uniform grid of values
Interpolation Methods – Many!

- All address the meaning of “near” in Tobler’s law differently
  - How does space make a difference?
  - Statistical mean not best predictor if Tobler’s law is true
Interpolation methods

- **Inverse Distance Weighting (IDW)**
  - Assumes influence of adjacent points decreases with distance

  \[
  z_0 = \frac{\sum_{i=1}^{n} w_i z_i}{\sum_{i=1}^{n} w_i}
  \]

  Where:

  - \( z_0 \) = value of estimation point
  - \( z_i \) = value of neighboring point
  - \( w_i \) = weighting factor; e.g. \( = 1/(\text{distance from neighbor})^2 \)
Inverse Distance Weighting

On basis of four nearest neighbors:

\[ z_0 = \frac{8/(1)^2 + 8/(2)^2 + 6/(2.5)^2 + 5/(2)^2}{1.66} \]

\[ z_0 = \frac{8.0 + 2.0 + 0.96 + 1.25}{1.66} = 7.36 \]
Inverse Distance Weighting (I.D.W.)

- Unknown value is the average of the observed values, weighted by inverse of distance, squared
  - Distance to point doubles, weight decreases by factor of 4

- Can alter IDW by:
  - Alter number of closest points
  - Choose points by distance/search radius
  - Weight be directional sectors
  - Alter distance weighting; e.g. cube instead of square
I.D.W. Characteristics

- Is an *Exact Method* of interpolation – will return a *measured value* when applied to measured point.
  - Will not generate smoothness or account for trends, unlike methods that are “inexact”
  - Answer (a surface) passes through data points

- Weights never negative → interpolated values can never be less than smallest z or greater than largest z. “Peaks” and “pits” will never be represented.
I.D.W. Characteristics

- No “peaks” or “pits” possible; interpolated values must lie within range of known values

![Diagram showing I.D.W. characteristics with trend surface, data points, and regions of misfit.]
Interpolation Methods

- IDW is inappropriate for values that don’t decrease as a function of distance (e.g. topography)

- Other deterministic, exact methods:
  - Spline
  - Natural Neighbor
Exact Methods - Spline

- Fit *minimum curvature* surface through observation points; interpolate value from surface
- Good for gently varying surfaces
  - E.g. topography, water table heights
- Not good for fitting large changes over short distances
- Surface is allowed to exceed highest and be less than lowest measured values
Exact Methods: IDW vs. Spline

IDW:
- No predicted highs or lows above max. or min. values
- No smoothing; surface can be rough

Spline:
- Minimum curvature result good for producing smooth surfaces
- Can’t predict large changes over short distances

(images from ArcGIS 9.2 Help files)
Comparisons - I.D.W. vs. Spline

- Note smoothing of Spline – less “spikey”
- IDW contours less continuous, fewer inferred maxima and minima

IDW, 6 nearest, contoured for 6 classes

Spline, contoured for same 6 classes
Inexact (Approximate) Methods

Inexact = Answer (surface) need not pass through input data points

- Trend surface – curve fitting by least squares regression
  - Deterministic – one output, no randomness allowed
- Kriging – weight by distance, consider trends in data
  - Stochastic – incorporates randomness
Approximate Methods - Trend

- Fits a polynomial to input points using least squares regression.
- Resulting surfaces minimize variance w.r.t. input values, i.e. sum of difference between actual and estimated values for all inputs is minimized.
- Surface rarely goes through actual points
- Surface may be based on all data (“Global” fit) or small neighborhoods of data (“Local” fits).
Trend Surfaces

Equations are either:

- **Linear – 1\textsuperscript{st} Order:** fit a plane
  - \( Z = a + bX + cY \)

- **Quadratic – 2\textsuperscript{nd} Order:** fit a plane with one bend (parabolic)
  - \( Z = (1\textsuperscript{st} \text{ Order}) + dX^2 + eXY + fY^2 \)

- **Cubic – 3\textsuperscript{rd} Order:** fit a plane with 2 bends (hyperbolic)
  - \( Z = (2\textsuperscript{nd} \text{ Order}) + gX^3 + hX^2Y + iXY^2 + Y^3 \)

Where:
- \( a, b, c, d, \text{ etc.} \) = constants derived from solution of simultaneous equations
- \( X, Y \) = geographic coordinates
Trend Surfaces – “Global Fitting”

Contour maps of trend surfaces for Z

Input data (x, y, z)

Linear (Plane)

Quadratic (Parabolic surface)

Cubic (Hyperbolic surface)

Source: Burrough, 1986
Trend Surfaces – Local fitting

- Local Polynomial Interpolation fits many polynomials, each within specified, overlapping “neighborhoods”.
- Neighborhood surface fitting is iterative; final solution is based on minimizing RMS error.
- Final surface is composed of best fits to all neighborhoods.
- Can be accomplished with tool in ESRI Geostatistical Analyst extension.
Trend Surfaces – Local fitting

- 2-D profile view of a model surface
  - Neighborhood 1 points (red) are being fit to a plane by iteration (2 steps are shown) and an interpolated point is being created

(images from ArcGIS 9.2 Help files)
Trend Surfaces – Local fitting, Step 2

- Model surface generated by many local fits
  - Note that several neighborhoods share some of the same data points: neighborhoods overlap

(images from ArcGIS 9.2 Help files)
Trend Surfaces – Local fitting, Step 3

Five different polynomials generate five local fits; in this example all are 1st Order.
Trend Surfaces – Local fitting, Step 4

- Result:

  (image from ArcGIS 9.2 Help files)

  Original data points are black
  Interpolated points are in colors

- Note that model surface (purple) passes through interpolated points, not measured data points.
Why Trend, Spline or IDW Surfaces?

- No strong reason to assume that z correlated with x, y in these simple ways
- Fitted surface doesn’t pass through all points in Trend
- *Data aren’t used to help select model*
- → Exploratory, *deterministic* techniques, but theoretically weak
Deterministic vs. Geostatistical Models

- Deterministic: purely a function of distance
  - No associated uncertainties are used or derived
  - E.g. IDW, Trend, Spline

- Geostatistical: based on statistical properties
  - Uncertainties incorporated and provided as a result
  - Kriging
Approximate Methods - Kriging

- Kriging
  - Another inverse distance method
  - Considers distance, cluster and spatial covariance (autocorrelation) – look for patterns in data
  - Fit function to selected points; look at correlation, covariance and/or other statistical parameters to arrive at weights – interactive process
  - Good for data that are spatially or directionally correlated (e.g. element concentrations)
Kriging

- Look for patterns over distances, then apply weights accordingly.

Steps:
1) Make a description of the spatial variation of the data - variogram
2) Summarize variation by a function
3) Use this model to determine interpolation weights
Kriging – Step 1

- Describe spatial variation with **Semivariogram**

\[ \frac{(Z_i - Z_j)^2}{2} \]

“Point cloud”

Distance between pairs of points
Kriging – Step 1

- Divide range into series of “lags” (“buckets”, “bins”)
- Find mean values of lags

\[(Z_i - Z_j)^2 / 2\]
Kriging – Step 2

- Summarize spatial variation with a function
  - Several choices possible; curve fitting defines different types of Kriging (circular, spherical, exponential, gaussian, etc.)

\[(Z_i - Z_j)^2 / 2\]
Kriging – Step 2

Key features of fitted variogram:

- **Nugget**: Semivariance at $d = 0$
- **Range**: $d$ at which semivariance is constant
- **Sill**: Constant semivariance beyond the range

![Variogram Diagram]

Distance between pairs of points ($d$) vs. Semivariance.
Kriging – Step 2

- Key features of fitted variogram:
  - Nugget – Measure of uncertainty of z values; precision of measurements
  - Range – No structure to data beyond the range; no correlation between distance and z beyond this value
  - Sill – Measure of the approximate total variance of z
Kriging – Step 2

Model surface profiles and their variograms:

- As local variation in surface increases, range decreases, nugget increases

Source: O’Sullivan and Unwin, 2003
Kriging – Step 3

- Determine Interpolated weights
  - Use fitted curve to arrive at weights – not explained here; see O’Sullivan and Unwin, 2003 for explanation
  - In general, nearby values are given greater weight (like IDW), but direction can be important (e.g. “shielding” can be considered)
Review:
Deterministic vs. Geostatistical Models

- **Deterministic**: interpolation purely a function of distance
  - No associated uncertainties are used or derived
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- **Geostatistical**: interpolation is statistically based
  - Uncertainties incorporated and provided as a result
    - Kriging