

10-5-2008

P. ①

## Supplementary Materials for Ch. 4 and Ch. 5

### Moisture Parameters

The presence of water vapor in the air may be expressed in many different ways. For examples, we have talked about the relative humidity  $RH$ , the dew point temperature  $T_d$ , among others.

Below is a list of moisture variables that are important in atmospheric sciences.

specific humidity	$q \equiv \frac{m_v}{m_d + m_v}$	$[kg\ kg^{-1}]$
mixing ratio	$w \equiv \frac{m_v}{m_d}$	$[kg\ kg^{-1}]$
vapor pressure	$e$	$[mb]$
dew point	$T_d$	$[^{\circ}C]$ or $[K]$ or $[^{\circ}F]$
relative humidity	$RH$	$[\%]$

where  $m_v$  = the mass of water vapor  
 $m_d$  = the mass of dry air

\* Knowing any one of these variables, plus the total pressure  $P$  and temperature  $T$ , other variables can be derived.

\* When a parcel of air rises in the atmosphere its temperature decreases with altitude at the dry adiabatic lapse rate until the air becomes saturated with water vapor. Further lifting follows the saturated-adiabatic lapse rate. Do you know how each of the above five moisture variables change with altitude?

$$q \rightarrow w = \frac{q}{1-q}$$

$q$  must be in  $\text{kg kg}^{-1}$   
 (In nature, often  $q \leq 0.02 \text{ kg kg}^{-1}$ )  
 $w$  must be in  $\text{kg kg}^{-1}$

$$w \rightarrow q = \frac{w}{1+w}$$

$$q \rightarrow e = \frac{qP}{0.622 + 0.378q}$$

$q$  must be in  $\text{kg kg}^{-1}$

$$w \rightarrow e = \frac{wP}{w+0.622}$$

$w$  must be in  $\text{kg kg}^{-1}$

see GPC Fig. 1.9  
 $0 \leq q \leq 0.02 \text{ kg kg}^{-1}$

$$e \rightarrow q = \frac{0.622 e}{p - 0.378 e}$$

(exact)

see GPC Fig. 1.8

$$\approx \frac{0.622 e}{p}$$

(approximate)  
 ( $\because p \gg e$ )

$0 \leq e \leq 17.5 \text{ mb}$

Decrease rapidly with altitude  
 and  $< 10\%$  of its  
 surface value at  $5 \text{ km}$ .

$$e \rightarrow w = 0.622 \frac{e}{p - e}$$

$T \rightarrow$

$$e^* = 6.11 \exp \left\{ \frac{L}{R_v} \left( \frac{1}{T_f} - \frac{1}{T} \right) \right\} \quad (\text{see B.3 in GPC})$$

[mb]

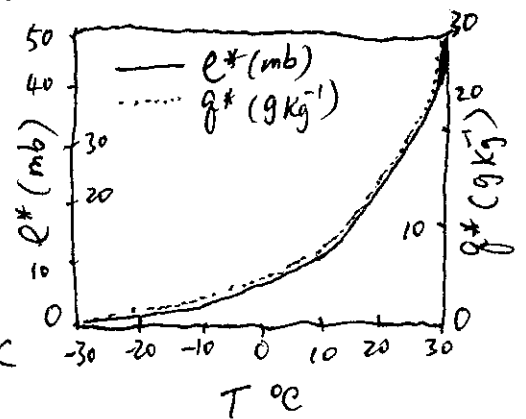
where  $L = 2.5 \times 10^6 \text{ J kg}^{-1}$  at  $0^\circ\text{C}$   
 ( $2.25 \times 10^6 \text{ J kg}^{-1}$  at  $100^\circ\text{C}$ )  
 latent heat of vaporization

$R_v = 461 \text{ J K}^{-1} \text{ kg}^{-1}$   
 gas constant for vapor

$T_f = 273.15 \text{ K}$   
 freezing point

$T = \text{air temperature in K}$

$$RH = \frac{e}{e^*} \times 100 = \frac{e}{e^*(T)} \times 100$$



$$e^* \rightarrow q^* = \frac{0.622 e^*}{p - 0.378 e^*}$$

$$\left( \doteq \frac{0.622 e^*}{p} \right)$$

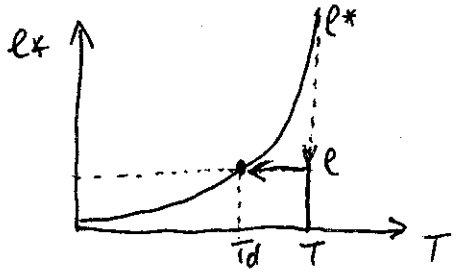
$$\frac{\partial q^*}{\partial T} = q^* \frac{L}{R_v T^2}$$

(Do you know how this is derived?)

It is not recommended to use this approximate equation in numerical weather prediction. Why?

$$e^* \rightarrow w^* = \frac{0.622 e^*}{p - e^*}$$

How to obtain  $T_d$ ?



The dew point  $T_d$  is the temperature to which air must be cooled at constant pressure in order for it to become saturated with respect to a plane surface of water.

In other words, the dew point is the temperature at which the saturation mixing ratio  $w^*$  with respect to water becomes equal to the actual mixing ratio  $w$ .

$$w = w^*(T_d) \quad \text{or}$$

$$q = q^*(T_d) \quad \text{or}$$

$$e = e^*(T_d)$$

Do you know where 0.622 is from?

$$\epsilon \doteq 0.622$$

$$\epsilon \equiv \frac{M_v}{M_d} = \frac{18}{29} \doteq 0.622$$

$M_v$ : molecular weight of water  
 $M_d$ : apparent molecular weight of dry air

$$\epsilon \equiv \frac{M_v}{M_d} \equiv \frac{R_d}{R_v}$$

$R_d = 287 \text{ J kg}^{-1} \text{ K}^{-1}$  gas constant of dry air

$R_v = 461 \text{ J kg}^{-1} \text{ K}^{-1}$  gas constant of water vapor

$$e = \frac{m_v/M_v}{m_d/M_d + m_v/M_v} P = \frac{w}{w + \epsilon} P$$

$$w = \frac{m_v}{m_d} = \frac{m_v/V}{m_d/V} = \frac{p_v}{p_d}$$

$$= \frac{e/(R_v T)}{(p - e)/(R_d T)} = \frac{R_d}{R_v} \frac{e}{p - e}$$

$$\doteq 0.622 \frac{e}{p - e} \approx 0.622 \frac{e}{p}$$

Knowing  $p, T, q \rightarrow$  using  $e = \frac{q p}{0.622 + 0.378 q}$  to obtain  $e$ ,  $\rightarrow$  then, let

$e$  be  $e^*$ , using  $e^* = 6.11 \exp\left\{\frac{L}{R_v} \left(\frac{1}{T^*} - \frac{1}{T}\right)\right\}$  to find  $T \rightarrow$  This  $T$  is  $T_d$ .

In addition to the above equation for  $e^*$ , another commonly used equation (e.g. in land surface modeling) is

$$e^* = 6.11 \exp\left[ \frac{A(T - T_f)}{T - B} \right] \quad \begin{matrix} [\text{mb}] \\ T \text{ in } [K] \end{matrix}$$

$$A = \begin{cases} 21.874 & T \leq T_f \\ 17.269 & T > T_f \end{cases}$$

$$B = \begin{cases} 7.66 & T \leq T_f \\ 35.86 & T > T_f \end{cases}$$

$$T_f = 273.15 \text{ K}$$

$$\frac{\partial e^*}{\partial T} = \frac{A(T_f - B)}{(T - B)^2} e^*$$

$$\frac{\partial q^*}{\partial T} = \frac{0.622 P}{(P - 0.378 e^*)^2} \frac{\partial e^*}{\partial T}$$

If the high accuracy is needed (e.g. for numerical cloud models), a sixth-order polynomial is used.

$$e^* = a_1 + \sum_{n=2}^7 a_n (T - T_f)^{n-1} \quad \begin{matrix} \text{valid for} \\ -50^\circ \leq T \leq 50^\circ \text{C} \end{matrix}$$

Table 1 [see Curry & Webster, 1999 which is after Flatau et al., 1992]

Table 4.3 Coefficients of the sixth-order polynomial fits to saturation vapor pressure for the temperature range  $-50^\circ$  to  $50^\circ\text{C}$  for both liquid water and ice. (After Flatau et al., 1992.)

Coefficient	Liquid water	Ice
$a_1$	6.11176750	6.10952665
$a_2$	0.443986062	0.501948366
$a_3$	0.143053301E-01	0.186288989E-01
$a_4$	0.265027242E-03	0.403488906E-03
$a_5$	0.302246994E-05	0.539797852E-05
$a_6$	0.203886313E-07	0.420713632E-07
$a_7$	0.638780966E-10	0.147271071E-09

Ideal gas law

$$PV = nR^* T$$

Both sides divided by  $m$ .

$$\frac{PV}{m} = \frac{nR^*}{m} T$$

$$P\alpha = RT$$

or  $\boxed{P = \rho R T}$

A parcel of air, ( $V, T, P, m, N$ )



- $V =$  volume  $[m^3]$
- $P =$  pressure  $[Pa]$
- $T =$  temperature  $[K]$
- $m =$  mass  $[kg]$
- $N =$  number of molecules

where  $\alpha = \frac{1}{\rho} = \frac{V}{m}$

$$R = \frac{nR^*}{m} = \frac{R^*}{(m/n)} = \frac{R^*}{M}$$

$R =$  gas constant  $[J kg^{-1} K^{-1}]$

$$R^* = \text{universal gas constant} = 8.31432 J K^{-1} mol^{-1} = N_0 k$$

$$k = \text{Boltzmann's constant} = 1.38 \times 10^{-23} J K^{-1}$$

$$n = \frac{N}{N_0} = \frac{\text{The number of molecules in } V}{\text{The number of molecules in a mole}}$$

$$\left. \begin{aligned} P &= P_d + e \\ P_d &= P_d R_d T \\ e &= P_v R_v T \end{aligned} \right\} \Rightarrow P = (P_d R_d + P_v R_v) T = \left( \frac{m_d}{V} R_d + \frac{m_v}{V} R_v \right) T$$

$$= \underbrace{\frac{m_d + m_v}{V}}_{\rho} \underbrace{\frac{m_d R_d + m_v R_v}{m_d + m_v}}_R T$$

$$\boxed{R = (1 - q) R_d + q R_v}$$

Note  $\epsilon = \frac{M_v}{M_d} \left( \frac{R_d}{R_v} \right) = \frac{18}{29} \approx 0.622$

$$= R_d \left[ 1 - q \left( \frac{1}{\epsilon} - 1 \right) \right] = \boxed{R_d (1 + 0.608 q)}$$

$$\therefore P = P R T = P R_d (1 + 0.608 q) T \quad T \text{ in } K, q \text{ in } kg kg^{-1}$$

Virtual  $T$ :  $T_v = (1 + 0.608 q) T \Rightarrow P = P R_d T_v$   
 interpreted as the  $T$  of dry air having the same values of  $P$  and  $\rho$  as the moist air.

⊛ Can you show that

$T_v$  is a function of  $P$ ,  $T$ , and  $RH$ , i.e.,

$$T_v = \frac{T}{1 - \frac{RH \cdot e^*(T)}{100 \cdot P} (1 - \epsilon)}$$

⊛ When  $RH=0$ , i.e., the dry air,  $P = P_{Rd} T$

⊛ If the air is not dry,  $P = P_{Rd} T_v$

In practice,  $q < 0.02 \text{ kg kg}^{-1}$ ,

$0.608 q T < 2 \text{ or } 3^\circ\text{C}$

$$\begin{aligned} \theta &= T \left(\frac{P_0}{P}\right)^{R_d/c_p} \\ \theta_v &= T_v \left(\frac{P_0}{P}\right)^{R_d/c_p} \\ &= \theta (1 + 0.608 q) \end{aligned}$$

Specific heats

specific heat at constant volume,  $C_v$

$$C_v \equiv \left(\frac{dq}{dT}\right)_{\alpha \text{ constant}} = \frac{du}{dT}$$

Specific heat at constant pressure,  $C_p$

$$C_p \equiv \left(\frac{dq}{dT}\right)_{P \text{ constant}}$$

First Law of Thermodynamics for an ideal gas ( $P\alpha = R_d T$ )

$$\begin{aligned} dq &= du + dw \\ &= C_v dT + P d\alpha \\ &= C_v dT + d(P\alpha) - \alpha dp \\ &= (C_v + R_d) dT - \alpha dp \end{aligned}$$

$$\therefore dq = C_{p_d} dT - \alpha dp$$

where

$$C_{p_d} = C_v + R_d$$

all in  $\text{J kg}^{-1} \text{K}^{-1}$

heat

$dq$

$\Rightarrow$



a unit of <sup>mass</sup> material

$\Rightarrow dt$

For dry air,

$$C_p = 1004 \text{ J kg}^{-1} \text{ K}^{-1}$$

$$C_v = 717 \text{ J kg}^{-1} \text{ K}^{-1}$$

$$R_d = 287 \text{ J kg}^{-1} \text{ K}^{-1}$$

For water  $C_p = 4218 \text{ J kg}^{-1} \text{ K}^{-1}$  (at  $0^\circ\text{C}$ )

Soil inorganic material  $C_p = 733 \text{ J kg}^{-1} \text{ K}^{-1}$

Soil organic material  $C_p = 1921 \text{ J kg}^{-1} \text{ K}^{-1}$

see also Table 4.1 in GPC and Appendix H.

The heat capacity of the entire atmosphere, [see GPC p.84]

$$\bar{C}_a = C_p \frac{p_s}{g} = \frac{1004 \text{ J kg}^{-1} \text{ K}^{-1} \cdot 10^5 \text{ Pa}}{9.81 \text{ ms}^{-2}} = 1.02 \times 10^7 \text{ J K}^{-1} \text{ m}^{-2}$$

If the entire atmosphere warms by  $2^\circ\text{C}$  in a 3-month period, the average rate of net energy flow into the atmosphere is

$$\bar{C}_a \frac{\Delta T}{\Delta t} = 1.02 \times 10^7 \text{ J K}^{-1} \text{ m}^{-2} \frac{2 \text{ K}}{365 \times 86400 \text{ s}/4}$$

$$= 2.59 \text{ W m}^{-2}$$

If the top 10 m of ocean warms by  $1^\circ\text{C}$  in a 3-month period, the average rate of net energy flow into the ocean is

$$\bar{C}_o \frac{\Delta T}{\Delta t} = \rho_w \cdot C_{pw} \cdot d_w \cdot \frac{\Delta T}{\Delta t} = 10^3 \text{ kg m}^{-3} \cdot 4218 \text{ J kg}^{-1} \text{ K}^{-1} \cdot 10 \text{ m} \frac{1 \text{ K}}{365 \times 86400 \text{ s}/4}$$

$$= 5.35 \text{ W m}^{-2}$$

Poisson's Equation:

First Law of Thermodynamics

$$dq = C_p dT - \alpha dp$$

For an adiabatic process ( $dq=0$ ), e.g. dry-adiabatic process

- (1) no moisture is present, or
- (2) no condensation of water vapor and no liquid water is present

$$C_p dT - \alpha dp = 0$$

Using an ideal gas law,  $\alpha = \frac{R_d T}{P}$

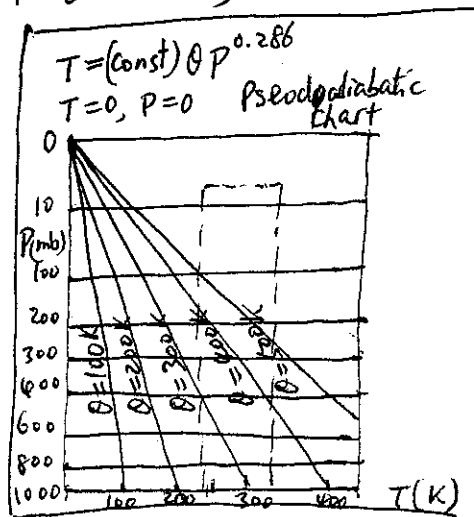
$$\frac{C_p}{R} \frac{dT}{T} - \frac{dp}{P} = 0$$

Integrating upward from  $P_0$  (where  $T=\theta$ ) to  $P$ ,

$$\frac{C_p}{R} \int_{\theta}^T \frac{dT}{T} = \int_{P_0}^P \frac{dp}{P}$$

or 
$$\frac{C_p}{R} \ln \frac{T}{\theta} = \ln \frac{P}{P_0}$$

$$\left(\frac{T}{\theta}\right)^{\frac{C_p}{R}} = \frac{P}{P_0}$$



or 
$$\theta = T \left(\frac{P_0}{P}\right)^{R/C_p}$$

→ Poisson's equation

Potential temperature;  $P_0 = 1000 \text{ mb}$   
 For dry air,  $R \doteq R_d = 287 \text{ J Kg}^{-1} \text{ K}^{-1}$ ,  $C_p \doteq C_{p,d} = 1004 \text{ J Kg}^{-1} \text{ K}^{-1}$

$$\therefore R/C_p \doteq 0.286$$

① is extremely useful in atmospheric thermodynamics.

① Every air parcel has a unique  $P, T$  and therefore ①

② ① is conserved for dry adiabatic motions.

③ ① is quasi-conserved for synoptic scale motions outside precipitation area!



Potential temperature

$$\theta = T \left( \frac{P_0}{P} \right)^{\frac{R_d}{C_p}}$$

Taking the logarithm of both sides,

$$\ln \theta = \ln T + \frac{R_d}{C_p} \ln P_0 - \frac{R_d}{C_p} \ln P$$

Differentiating with time (t):

$$C_p \frac{d \ln \theta}{dt} = C_p \frac{d \ln T}{dt} - R_d \frac{d \ln P}{dt}$$

$$C_p \frac{d \ln \theta}{dt} = \frac{ds}{dt} \quad \left[ \text{Note } \boxed{ds \equiv \frac{dq}{T} = C_p d \ln T - R_d d \ln P} \right]$$

A parcel that conserves entropy must move along a constant  $\theta$  surface (isentropic surface).

Dry adiabatic lapse rate:

- ①  $dq = C_p dT - \alpha dp$  (1<sup>st</sup> law of thermodynamics)
- ②  $\alpha dp = -g dz$  (The hydrostatic equation)
- ③  $dq = 0$  (Adiabatic)
- ④  $C_p dT + g dz = 0$

$$\Gamma_d \equiv \frac{g}{C_p} = - \frac{dT}{dz} \Big|_{\text{adiabatic}}$$

$$= 9.8 \text{ } ^\circ\text{C/km}$$

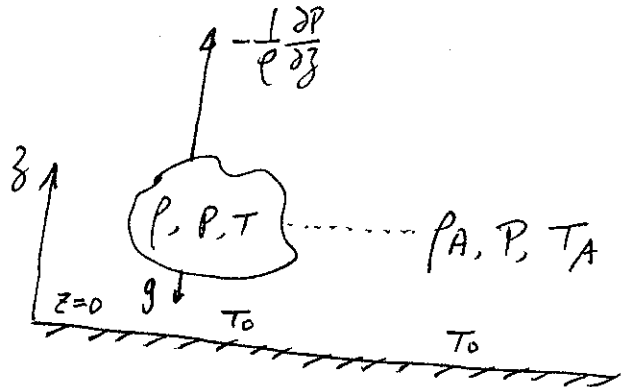
# Linking the lapse rate with the vertical motion

p10

For the air parcel,  $\frac{dw}{dt} = -g - \frac{1}{\rho} \frac{\partial p}{\partial z}$

For the environment,  $0 = -g - \frac{1}{\rho_A} \frac{\partial p}{\partial z}$

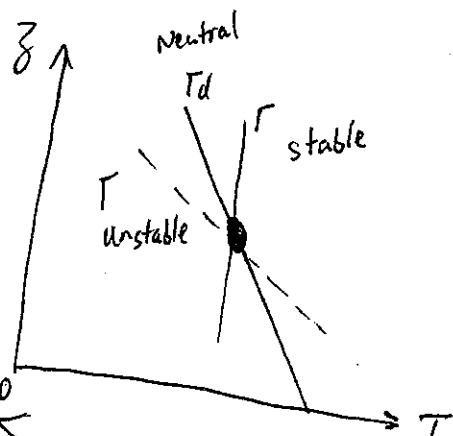
$$\begin{aligned} \frac{dw}{dt} &= \frac{\partial p}{\partial z} \left( \frac{1}{\rho_A} - \frac{1}{\rho} \right) \\ &= \frac{\partial p}{\partial z} \left( \frac{\rho - \rho_A}{\rho \rho_A} \right) = \frac{g(\rho_A - \rho)}{\rho} = g \frac{\frac{\rho}{RT_A} - \frac{\rho}{RT}}{\frac{\rho}{RT}} \\ &= \frac{g(T - T_A)}{T_A} \end{aligned}$$



Assuming  $T(z) = T_0 - \Gamma_d z$   
 $T_A(z) = T_0 - \Gamma z$

$$\frac{dw}{dt} = \frac{g z}{T_A} (\Gamma - \Gamma_d)$$

$\frac{dw}{dt} <$	stable	$\Gamma <$	$\Gamma_d$
$\frac{dw}{dt} = 0$	neutral	$\Gamma =$	$\Gamma_d$
$\frac{dw}{dt} >$	unstable	$\Gamma >$	$\Gamma_d$



$$\frac{dw}{dt} = -\frac{g z}{\theta} \frac{\partial \theta}{\partial z} \quad (\text{see next page})$$

Relationship b/n  $\frac{\partial \theta}{\partial z}$  and  $\frac{\partial T}{\partial z}$  :

$$\theta = T \left( \frac{P_0}{P} \right)^{\frac{R}{c_p}}$$

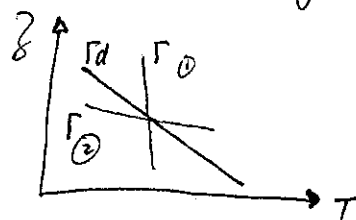
$$\ln \theta = \ln T + \frac{R}{c_p} \ln P_0 - \frac{R}{c_p} \ln P$$

$$\frac{\partial}{\partial z} \text{ both sides: } \frac{1}{\theta} \frac{\partial \theta}{\partial z} = \frac{1}{T} \frac{\partial T}{\partial z} - \frac{R}{c_p} \frac{1}{P} \frac{\partial P}{\partial z}$$

using hydrostatic eqn and ideal gas law

$$\frac{T}{\theta} \frac{\partial \theta}{\partial z} = \frac{\partial T}{\partial z} + \frac{g}{c_p}$$

$$\boxed{\frac{T}{\theta} \frac{\partial \theta}{\partial z} = \Gamma_d - \Gamma}$$



①  $\frac{\partial \theta}{\partial z} > 0$ ,  $\Gamma_d > \Gamma$ , the parcel colder than the environment, sink to the original, stable

②  $\frac{\partial \theta}{\partial z} < 0$ ,  $\Gamma_d < \Gamma$ , the parcel warmer than the environment, rise by buoyancy, unstable

③  $\frac{\partial \theta}{\partial z} = 0$ ,  $\Gamma_d = \Gamma$  neutral

$$\frac{dw}{dt} = -\frac{g}{\theta} \frac{\partial \theta}{\partial z} \quad w = \frac{dz}{dt} \quad \boxed{\frac{d^2 z}{dz^2} + \left( \frac{g}{\theta} \frac{\partial \theta}{\partial z} \right) z = 0}$$

2<sup>nd</sup> order ordinary differential eqn.

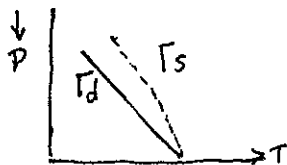
The solution is a sinusoidal variation in  $z$  with

a frequency ( $\text{rad s}^{-1}$ )  $N = \left( \frac{g}{\theta} \frac{\partial \theta}{\partial z} \right)^{\frac{1}{2}}$

If  $\Gamma_d > \Gamma$ ,  $N$  is real, the solution is  $z = A \cos Nt + B \sin Nt$

The Brunt-Väisälä frequency

Saturated-adiabatic process:



①  $dQ = c_p dT + g dz$ ; ②  $dQ = -L dq^*$

③  $\frac{\partial q^*}{\partial z} = \frac{\partial q^*}{\partial T} \cdot \frac{\partial T}{\partial z}$  ④  $\Gamma_d = \frac{g}{c_p}$

$$\therefore \Gamma_s \equiv - \frac{dT}{dz} \Big|_{\text{saturated-adiabatic}} = \frac{\Gamma_d}{1 + \frac{1}{c_p} \frac{\partial q^*}{\partial T}}$$

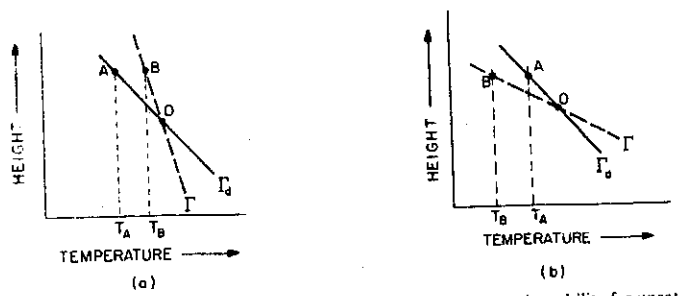
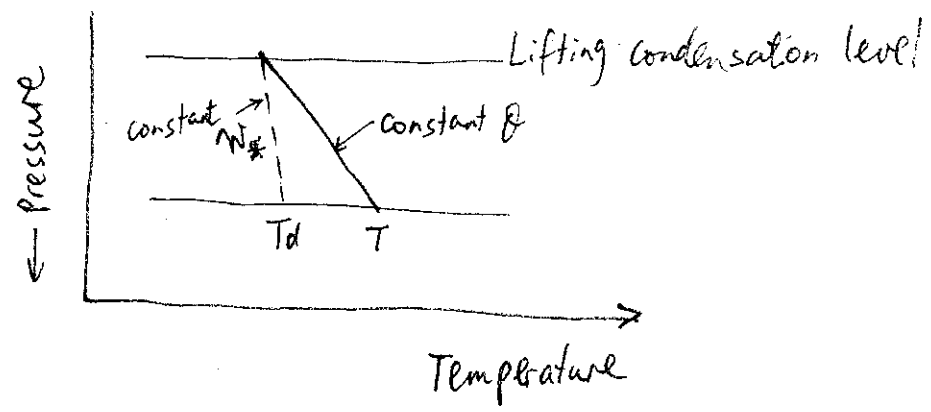


Fig. 2.10 Conditions for (a) positive static stability and (b) negative static stability for unsaturated air. Negative static stability can only exist very close to the ground.

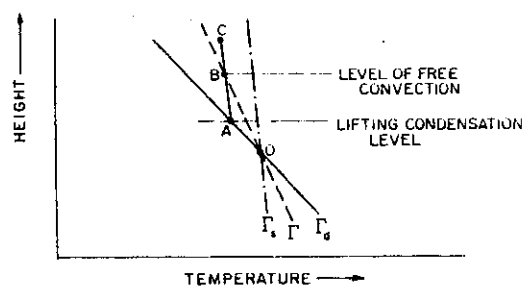


Fig. 2.12 Conditions for conditional instability.  $\Gamma_s$  and  $\Gamma_d$  are the saturated and dry adiabatic lapse rates and  $\Gamma$  the actual lapse rate of temperature of the air.

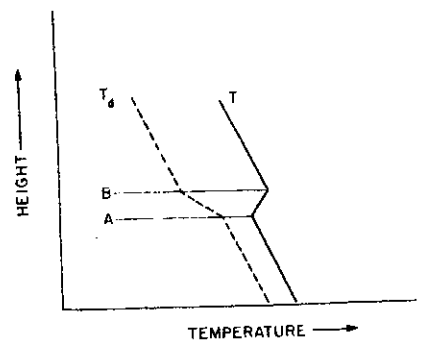
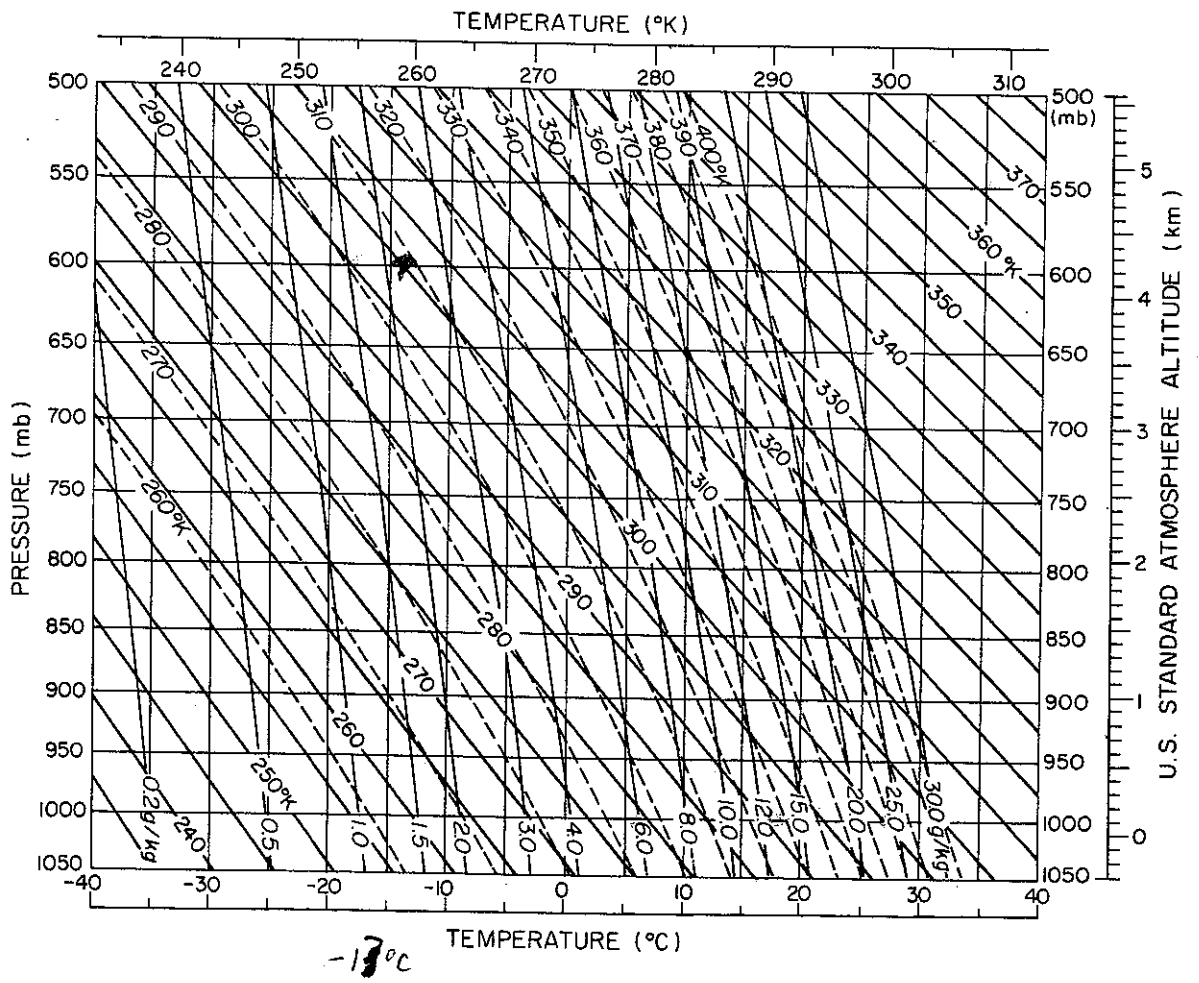
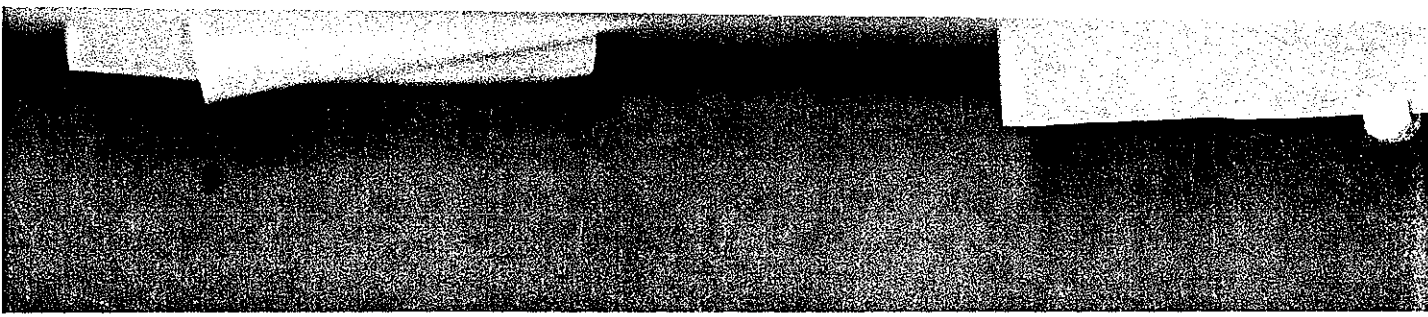


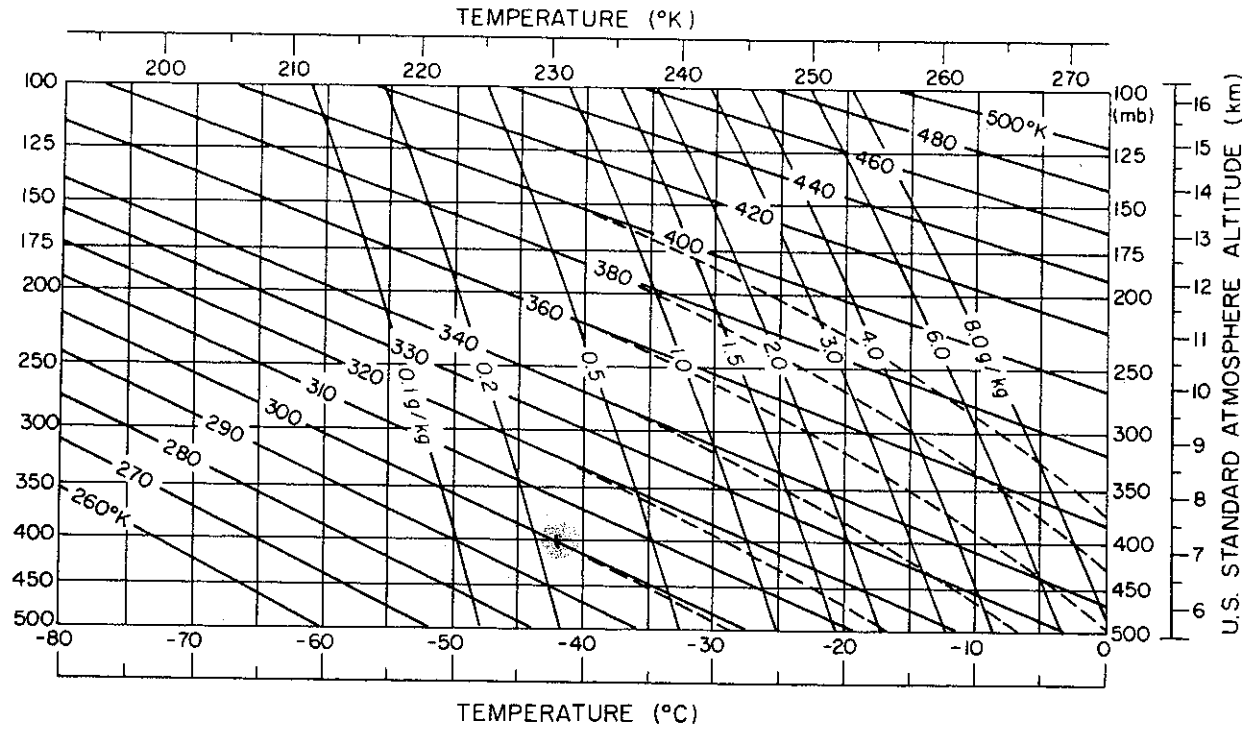
Fig. 2.13 Conditions for convective instability.  $T$  and  $T_d$  are the temperature and dew point of the air



-17°C



The pseudoadiabatic chart Black sloping lines denote *dry adiabats* which are lines of constant potential temperature  $\theta$  labeled in degrees Kelvin. Red solid lines refer to saturation mixing ratio  $w_s$ , labeled in grams of water vapor per kilogram of dry air. Dashed red lines denote *pseudoadiabats*, which are lines of constant equivalent potential temperature  $\theta_e$ , for saturated air, labeled in degrees Kelvin. The ordinate scale is linearly proportional to  $p^{R_d/c_p}$ . The altitude scale at the right is for the U.S. Standard Atmosphere. See text for further explanation.



$$\theta = T \left( \frac{P_0}{P} \right)^{\frac{R_d}{c_p}}$$

0.286

$$T = (\text{const}) \theta P$$